Engineering Notes

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Simple Computational Technique for Determining Angle-of-Attack Convergence of Conical Re-entry Vehicles

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Nomenclature

$C_{m\alpha} = \frac{\partial C_m}{\partial \alpha}$, $C_{mq} = \frac{\partial C_m}{\partial q}$, $C_{m\dot{\alpha}} = \frac{\partial C_m}{\partial \dot{\alpha}}$, $C_{m_{p\beta}} = \frac{\partial^2 C_m}{\partial \rho \partial \beta}$ } stability derivatives

d = base diameter
I_x = moment of inertia about longitudinal axis
I = moment of inertia about yaw and pitch axes
P = re-entry vehicle spin rate along longitudinal axis
Q = dynamic pressure ($\frac{1}{2}\rho V^2$)
q = pitch rate
S = reference area of re-entry vehicle, usually area of base
V = freestream velocity
α_t = pitch angle of attack at time $= t$
$\dot{\alpha}_t$ = $\frac{\partial \alpha_t}{\partial t}$
β_t = yaw angle of attack at time $= t$
$\dot{\beta}_t$ = $\frac{\partial \beta_t}{\partial t}$
α_{t-1},β_{t-1} = yaw and pitch at time $t-\Delta t$
Δt = time increment
ω_1,λ_1 = nutational frequency and damping term
ω_2,λ_2 = precessional frequency and damping term

Introduction

THE typical solution to the equations of motion that govern the pitching and yawing motion of an axisymmetric re-entry vehicle require the solution of Bessel functions of the first and second kind. This approach can be useful if the atmosphere density follows a strict exponential. Solutions can become much more complicated if the atmospheric profile does not follow a simple function.

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The alternative approach that is provided here can be used whenever conditions of the re-entry are not defined by simple functions. The method uses the solutions for a constant density atmosphere to compute very accurate angular velocities and positions over small time increments (about 0.04 s).

The angle-of-attack solution for a constant density atmosphere is defined by Regan[1] (Eq. 10.47). For the technique described here, initial conditions are taken to be those at time $t-\Delta t$, and the time t is set equal to the time step Δt. Regan's solution is then redefined as follows:

$$\alpha_t = [A_1 \cos(\omega_1 \Delta t) + B_1 \sin(\omega_1 \Delta t)]e^{\lambda_1 \Delta t} + [A_2 \cos(\omega_2 \Delta t) + B_2 \sin(\omega_2 \Delta t)]e^{\lambda_2 \Delta t}$$

$$\beta_t = [B_1 \cos(\omega_1 \Delta t) - A_1 \sin(\omega_1 \Delta t)]e^{\lambda_1 \Delta t} + [B_2 \cos(\omega_2 \Delta t) - A_2 \sin(\omega_2 \Delta t)]e^{\lambda_2 \Delta t}$$

where

$$A_1 = [(\dot{\alpha}_{t-1} - \lambda_2 \alpha_{t-1} - \omega_2 \beta_{t-1})(\lambda_1 - \lambda_2) - (\dot{\beta}_{t-1} - \lambda_2 \beta_{t-1} + \omega_2 \alpha_{t-1})(\omega_1 - \omega_2)] \div [(\omega_1 - \omega_2)^2 + (\lambda_1 - \lambda_2)^2]$$

$$A_2 = [(\dot{\alpha}_{t-1} - \lambda_1 \alpha_{t-1} - \omega_1 \beta_{t-1})(\lambda_2 - \lambda_1) - (\dot{\beta}_{t-1} - \lambda_1 \beta_{t-1} + \omega_1 \alpha_{t-1})(\omega_2 - \omega_1)] \div [(\omega_1 - \omega_2)^2 + (\lambda_1 - \lambda_2)^2]$$

$$B_1 = [(\dot{\alpha}_{t-1} - \lambda_2 \alpha_{t-1} - \omega_2 \beta_{t-1})(\omega_1 - \omega_2) - (\dot{\beta}_{t-1} - \lambda_2 \beta_{t-1} + \omega_2 \alpha_{t-1})(\lambda_1 - \lambda_2)] \div [(\omega_1 - \omega_2)^2 + (\lambda_1 - \lambda_2)^2]$$

$$B_2 = [(\dot{\alpha}_{t-1} - \lambda_1 \alpha_{t-1} - \omega_1 \beta_{t-1})(\omega_2 - \omega_1) - (\dot{\beta}_{t-1} - \lambda_1 \beta_{t-1} + \omega_1 \alpha_{t-1})(\lambda_2 - \lambda_1)] \div [(\omega_1 - \omega_2)^2 + (\lambda_1 - \lambda_2)^2]$$

$$\omega_1 = \frac{PI_x}{2I}\left(1 + \frac{1}{\tau}\right) \qquad \omega_2 = \frac{PI_x}{2I}\left(1 - \frac{1}{\tau}\right)$$

$$\lambda_1 = \left[\frac{QSd^2 (C_{mq} + C_{m\dot{\alpha}})}{4VI}\right](1+\tau) + \left[\frac{QSd^2 C_{m_{p\beta}}}{I_x}\right]\tau$$

$$\lambda_2 = \left[\frac{QSd^2 (C_{mq} + C_{m\dot{\alpha}})}{4VI}\right](1-\tau) - \left[\frac{QSd^2 C_{m_{p\beta}}}{I_x}\right]\tau$$

$$\tau = \frac{PI_x/2I}{\sqrt{(PI_x/2I)^2 - (QSdC_{m\alpha}/I)}}$$

Angular velocities are found by explicitly differentiating the equations for angle of attack to obtain

$$\begin{aligned}\dot{\alpha}_t &= \lambda_1[A_1 \cos(\omega_1 \Delta t) + B_1 \sin(\omega_1 \Delta t)]e^{\lambda_1 \Delta t} \\ &+ \omega_1[B_1 \cos(\omega_1 \Delta t) - A_1 \sin(\omega_1 \Delta t)]e^{\lambda_1 \Delta t} \\ &+ \lambda_2[A_2 \cos(\omega_2 \Delta t) + B_2 \sin(\omega_2 \Delta t)]e^{\lambda_2 \Delta t} \\ &+ \omega_2[B_2 \cos(\omega_2 \Delta t) - A_2 \sin(\omega_2 \Delta t)]e^{\lambda_2 \Delta t}\end{aligned}$$

$$\begin{aligned}\dot{\beta}_t &= \lambda_1[B_1 \cos(\omega_1 \Delta t) - A_1 \sin(\omega_1 \Delta t)]e^{\lambda_1 \Delta t} \\ &- \omega_1[A_1 \cos(\omega_1 \Delta t) + B_1 \sin(\omega_1 \Delta t)]e^{\lambda_1 \Delta t} \\ &+ \lambda_2[B_2 \cos(\omega_2 \Delta t) - A_2 \sin(\omega_2 \Delta t)]e^{\lambda_2 \Delta t} \\ &- \omega_2[A_2 \cos(\omega_2 \Delta t) + B_2 \sin(\omega_2 \Delta t)]e^{\lambda_2 \Delta t}\end{aligned}$$

Terms due to any trim moment have been ignored here, but the missing terms from Eq. 10.47 (Ref. 1) can simply be superimposed to account for the effects of a trim moment.

The velocity and altitude of the re-entry vehicle are easily determined using conventional methods that treat the vehicle as a point mass. There are various analytical methods for determining values for the stability derivatives, but the use of experimentally derived values was found to be both faster and more accurate. In general, one experimental value for C_{m_q} and $C_{m_{\dot{\alpha}}}$ valid below 200,000 ft is appropriate (see Ref. 1, Chap. 9). For C_{m_α}, however, there should be enough data so that accurate interpolations can be produced for all altitudes and angles of attack.

The solutions provided by this technique, which was implemented in a Fortran code named SAMAC (Simplified Algorithm to Model Aerodynamic Convergence), were found to compare closely to those of a 6-deg-of-freedom code named RAKOT (written for the Air Force under the Space and Missiles System Organization[2]). Figures 1a and 1b depict plots of the magnitude of angle of attack vs altitude for both methods. Even where there is a resonance phenomenon, differences between the two plots are barely discernible to the naked eye.

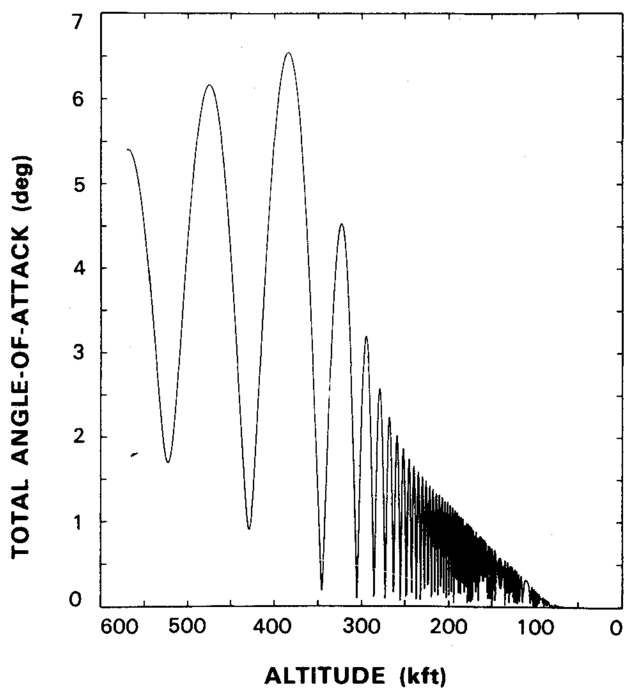

Fig. 1a Magnitude of angle of attack vs altitude for 6 DOF code.

A goodness of fit (GOF) parameter was conceived for plots of the angle of attack vs altitude. GOF was defined as:

$$1 = \frac{\text{area between curves of the average value determined by SAMAC and the average value determined by RAKOT}}{\text{area under the average value determined by RAKOT}}$$

where a GOF of 1.0 is a perfect match (averages were obtained by averaging the peak and base values of each cycle). Figure 2 shows the two curves. The area between them is hatched. For a typical re-entry with an initial angle of attack of 5.4 deg and a coning half angle of 3.9 deg, the GOF for the magnitude of the angle of attack was found to be 0.98 and the GOF for the rate of change of the angle of attack was 0.81. The run time

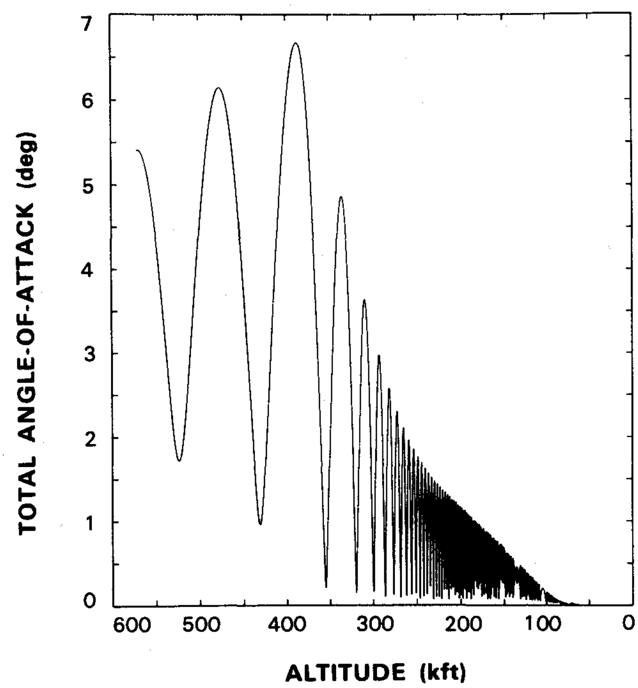

Fig. 1b Magnitude of angle of attack vs altitude for SAMAC.

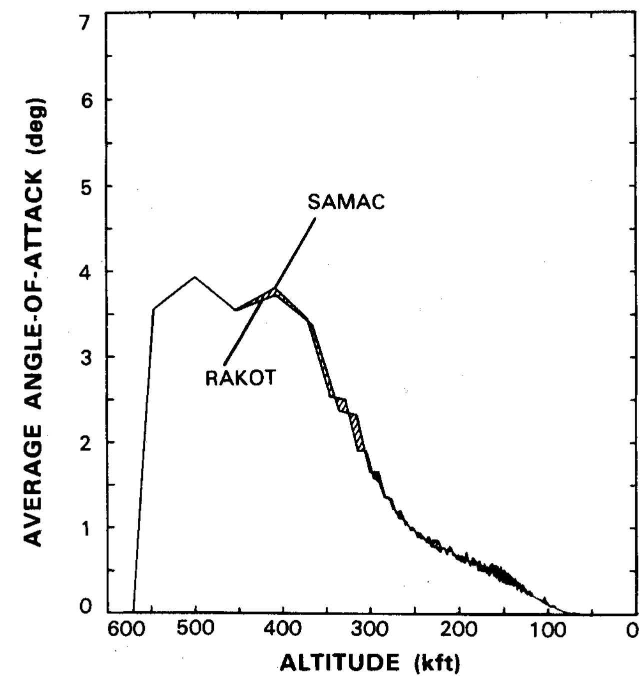

Fig. 2 Peak to base average values for magnitude of angle of attack for SAMAC and RAKOT. Hatched area represents discrepancies between the two solutions.

for this 500,000 ft re-entry was approximately 6 s on an IBM 3081 mainframe. The run time for the same re-entry computed by RAKOT was approximately 10 min.

Conclusions

This paper has explored a new method for computing the yawing and pitching motion of an axisymmetric re-entry vehicle. Several simulations indicate that the new method is both very accurate, in comparison to a full 6-deg-of-freedom computation, and very economical, requiring a factor of 80 times less CPU per simulation.

Acknowledgment

This work was sponsored by the Department of the Air Force. The views expressed are those of the author, and do not reflect the official policy or position of the U.S. Government.

References

[1]Regan, Frank J., *Re-entry Vehicle Dynamics*, AIAA Education Series, New York, 1984.

[2]Powel, F., "RAKOT 3D/6D Trajectory Simulation Program," Aerospace Corp., Aerospace Rept. TR-0074 (9320)-2, Aug. 1973.

Kalman Filtering for Second-Order Models

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I. Introduction

THIS paper develops Kalman filter algorithms for systems described by linear second-order matrix differential or difference equations, referred to as second-order models. Examples of such models arise frequently in many practical mechanical and aerodynamic systems and in the study of large space structures (LSS). Continuous-time models generally take the form

$$M\ddot{x}+D\dot{x}+Kx=Bu, \qquad x(0)=x_0, \qquad \dot{x}(0)=x_1 \tag{1a}$$

where, usually, M is symmetric positive definite and K is symmetric nonnegative definite. In this paper, we require only that M be invertible, although M^{-1} is not needed explicitly in our subsequent development. In some mechanical systems, for example, the constant matrices $M,D,K\epsilon R^{n\times n}$ are called the mass (or inertia), damping, and stiffness matrices, respectively, the state vector $x\epsilon R^n$ is called the displacement vector, and $F=Bu$ is called the force vector, where $u\epsilon R^m$ is a known input and $B\epsilon R^{n\times m}$ is a constant matrix.[1] In most situations, a set of measurements, $y\epsilon R^r$, rather than the full state vector x, is available, where

$$y=H_1x+H_2\dot{x} \tag{1b}$$

with $H_1,H_2\epsilon R^{r\times n}$. Various discrete-time versions of the above model can be found via sampling Eq. (1). By appropriately approximating $\dot{x}$ and $\ddot{x}$, we arrive at different discrete-time models. For example, $\dot{x}$ can be approximated by

$$\dot{x}\approx\frac{1}{h}\,[x(t+h)-x(t)] \tag{2a}$$

or by

$$\dot{x}\approx\frac{1}{h}\,[x(t)-x(t-h)] \tag{2b}$$

and $\ddot{x}$ can be approximated by one of the following:

$$\ddot{x}\approx\frac{1}{h^2}\,[x(t+2h)-2x(t+h)+x(t)] \tag{3a}$$

$$\ddot{x}\approx\frac{1}{h^2}\,[x(t+h)-2x(t)+x(t-h)] \tag{3b}$$

or

$$\ddot{x}\approx\frac{1}{h^2}\,[x(t)-2x(t-h)+x(t-2h)] \tag{3c}$$

where h is the sampling time. Using Eqs. (2) and (3) in Eq. (1a) and defining $t=kh$ and $x_k=x(kh)$, we obtain the following:

$$Mx_{k+2}+(-2M+hD)x_{k+1}+(M-hD+h^2K)x_k=h^2Bu_k \tag{4a}$$

$$Mx_{k+1}+(-2M+hD+h^2K)x_k+(M-hD)x_{k-1}=h^2Bu_k \tag{4b}$$

$$(M+hD)x_{k+1}+(-2M-hD+h^2K)x_k+Mx_{k-1}=h^2Bu_k \tag{4c}$$

$$(M+hD+h^2K)x_k+(-2M-hD)x_{k-1}+Mx_{k-2}=h^2Bu_k \tag{4d}$$

Equations (4a) and (4b) have the advantage that they keep the mass matrix intact, so that any property of M, such as symmetry or positive definiteness, is preserved.

In order to keep the notation as suggestive and yet as "clean" as possible in the sequel, we shall use the following generic discrete-time model:

$$Mx_{k+2}+Dx_{k+1}+Kx_k=Bu_k \tag{5a}$$

$$y_k=H_1x_k+H_2x_{k+1} \tag{5b}$$

with known initial conditions x_0 and x_1. By reference to Eq. (4), it is, of course, understood that the $M,K,D,\ldots$ matrices of Eq. (5) are not the same as the $M,K,D,\ldots$ matrices of Eq. (1).

We refer to the system in Eq. (1) or (5) as a second-order model. Although it might appear that the model in Eq. (5b) is not causal, this is essentially just a matter of notation. A first-order realization of a second-order model clearly has a standard (causal) observation equation. Or the apparent noncausality can be removed by a simple change of variables, e.g., by letting $\bar{y}_{k+1}=y_k$, which will easily be seen not to affect the subsequent development. In the sequel, we shall therefore work with the generic discrete-time model (5) (and its stochastic analog, to be described).

In practice, the measurements and the plant might be corrupted by noise, in which case a stochastic version of the second-order models in Eqs. (1) and (5) is used as follows:

$$M\ddot{x}+D\dot{x}+Kx=Bu+Gw \tag{6a}$$

$$z=y+v=H_1x+H_2\dot{x}+v \tag{6b}$$

Received Dec. 29, 1986; revision received March 30, 1987.
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